On Skew Class (BQ) Operators

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***Abstract-** The class of Skew (BQ) operators acting on the complex Hilbert space H is introduced in this paper. An operator if $K \in L(H)$ is said to belong to class Skew (BQ) if K commutes with a (BQ) operator, that is, $[K^{*2}K^2 (K^*K)^2] K = K [(K^*K)^2 K^{*2}K^2]$. We explore some properties that this class is enriched with. We then scan the relation of this class to other classes and then oversimplify it to the class of Skew (nBQ).*

***Indexed Terms-** Skew-Normal, Skew-Binormal operators, (BQ) operators, Skew-(BQ) Operators.*

I. INTRODUCTION

All the way through this paper, H implies the customary Hilbert space over the complex field and L (H) the Banach algebra of all bounded linear algebra on an infinite dimensional separable Hilbert space H. WanjalaVictor and Beatrice Adhiambo analyzed the class of (BQ) operators in (5). An operator $K \in L(H)$ is said to be class (BQ) if $K^{*2}K^2$ and $(K^*K)^2$ commutes. Various properties were covered including the relation between class (BQ) and class (Q) operators. We reverberate the findings of WanjalaVictor and Beatrice Adhiambo in (3) by introducing the class of skew (BQ) and look at the relation between this class and other classes and their general properties. An operator $K \in L(H)$ is allegedly class (Q) if $K^{*2}K^2 = (K^*K)^2$ (1). An operator $A \in L(H)$ is skew binormal if $(K^* K KK^*)K = K(K K^* K^*K)$ (2) . $K \in L(H)$ is n-skew n-binormal if $(K^*K^n K^nK^*)K^n = K^n(K^n K^* K^*K^n)$ (3). An operator $K \in L(H)$ is presumably Skew (BQ) if A commutes with a (BQ) operator, that is $[(K^*K)^2 (K^*K)^2] K = K [K^{*2}K^2 (K^*K)^2]$. We denote the class of Skew (BQ) by s(BQ).

II. MAIN RESULTS

- Theorem 1.1.If $K \in L(H)$ is such that $K \in s(BQ)$, then the subsequent are also proper for s(BQ);

(i). λK for any real λ

(ii). Any $B \in L(H)$ that is unitarily equivalent to K.

(iii). The restriction T-M to any closed subspace M of H.

Proof. (i). The proof is trivial.

(ii). Let $B \in L(H)$ be unitarily equivalent to K , then there exists a unitary operator $U \in L(H)$ with $K = U^*BU$ and $K^* = U^*B^*U$. $K \in s(BQ)$ implies ; $[K^{*2}K^2(K^*K)^2]K = K[(K^*K)^2K^{*2}K^2]$, hence

$[B^{*2}B^2(B^*B)^2]B = K[UK^{*2}U^*UK^2U^*(UK^*U^*UKU^*)^2]$

$= UK[K^{*2}U^*U^*K^2U^*UK^*U^*UK^*U^*UKU^*UK]U^*$

$= UK[(K)^{*2}K^2(K^*K)^2]U^*$

$= UK[(K^*K)^2K^{*2}K^2]U^*$

and

$[(B^*B)^2B^{*2}B^2]B = K[(UK^*U^*UKU^*)^2UK^{*2}U^*UK^2U^*]$

$= UK[K^*U^*UKU^*UK^*U^*UKU^*UK^{*2}U^*UK^2]U^*$

$= UK[K^*KK^*KK^{*2}K^{*2}]U^*$

$= UK[(K^*K)^2K^{*2}K^2]U^*$

Hence B is unitarily equivalent to K .

(iii) . If K is in s(BQ), then; $[K^{*2}K^2(K^*K)^2]K = K[(K^*K)^2K^{*2}K^2]$. Hence;

$[(K/M)^{*2}(K/M)^2\{(K/M)^*(K/M)\}^2]K/M$

$= [(K/M)^2(K/M)^2\{(K/M)^*(K/M)\}^2]K/M$

$= [(K^{*2}/M)(K^2/M)\{(K^*/M)(K/M)\}\{(K^*/M)(K/M)\}]K/M$

$= [((K^*K)^2/M)(K^{*2}K^2/M)]K/M$

$= [((K^*/M)(K/M))^2(K/M)^{*2}(K/M)^2]K/M$

Hence $K/M \in s(BQ)$.

- Theorem 2.If $K \in L(H)$ is in (BQ), then $K \in s(BQ)$.

Proof.

If $K\in$ (BQ), then $K^{*2}K^2 (K^*K)^2 = (K^*K)^2 K^{*2}K^2$
$(K^{*2}K^2 (K^*K)^2) K = K^{*2}K^2(K^*K)^2 K$

$= (K^*K)^2 K^{*2}K^2K$

$= K^{*2}K^2 K^{*2}K^2KK$

$= K^{*2}K^2 K^{*2} K K^2$

$= K^{*2}K^2 K K^{*2}K^2$

$= K^{*2} K K^2 K^{*2}K^2$

$= K K^{*2}K^2 K^{*2}K^2$

$= K ((K^*K)^2 K^{*2}K^2)$

As desired, hence the proof.

- Theorem 3.Let $K\in s(BQ)$ and $B \in s(BQ)$. $KB \in s(BQ)$ provided both K and B are doubly commuting.

  Proof.

$K\in s(BQ)$ implies ;

$(K^{*2}K^2 (K^*K)^2) K = K ((K^*K)^2 K^{*2}K^2)$

Similarly $B \in s(BQ)$ implies ;

$(B^{*2}B^2 (B^*B)^2) B = B ((B^*B)^2 B^{*2}B^2)$

$[(KB)^{*2}(KB)^2((KB)^*(KB))^2](KB) = (KB)[((KB)^*(KB))^2 (KB)^{*2}(KB)^2]$

$=K^{*2}B^{*2}K^2B^2((KB)^*(KB))((KB)^*(KB))(KB)$
$=K^{*2}B^{*2}K^2B^2((K^*B^*)(KB))((K^*B^*)(KB))(KB)$
$=K^{*2}B^{*2}K^2B^2K^*B^*KBK^*B^*KB K^*B^*KBKB$
$= K^{*2}B^{*2}K^2B^2K^*KB^*BK^*KB^*BKB$
$= B^{*2}B^2K^{*2}K^2B^*BB^*BK^*KK^*KKB$
$= B^{*2}B^2K^{*2}K^2(K^*K)^2B^*BB^*BKB$
$= B^{*2}B^2(K^*K)^2K^{*2}K^2B^*BB^*BKB$

$= KB B^{*2}B^2(K^*K)^2K^{*2}K^2B^*BB^*B$ (Since $K\in s(BQ)$).
$= KB (K^*K)^2 B^{*2}B^2B^*BB^*BK^{*2}K^2$
$= KB (K^*K)^2B^{*2}B^2(B^*B)^2K^{*2}K^2$
$= KB(K^*K)^2(B^*B)^2B^{*2}B^2K^{*2}K^2$ (Since $B \in s(BQ)$).
$= KB((K^*K)(B^*B))^2B^{*2}K^{*2}B^2K^2$
$= KB((K^*B^*)(KB))^2K^{*2}B^{*2}K^2B^2$
$= KB[((KB)^*(KB))^2(KB)^{*2}(KB)^2]$
Hence $KB \in s(BQ)$.

- Theorem 4.Let $K\in L(H)$ be such that it's both self-adjoint , 2-self-adjoint and be a s(BQ) operator , then $K^*\in s(BQ)$.

Proof.

$K\in s(BQ)$ implies ;

$[K^{*2}K^2(K^*K)^2] K = K [(K^*K)^2K^{*2}K^2]$

A being self-adjoint implies $K = K^*$ and 2-self-adjoint implies $K^2= K^{*2}$
Thus;

$[K^{*2}K^2(K^*K)^2]K = K^{*2}K^2(K^*K)^2 K$

$= K^2 K^{*2} (KK^*)^2K^*$

$= K^2 K^{*2} K^2 K^{*2} K^*$

$= K^4 K^{*5}$ ……………………(α)

$= K [(K^*K)^2K^{*2}K^2]$

$= K (K^*K)^2K^{*2}K^2$

$= K^* (KK^*)^2 K^2 K^{*2}$

$= K^* K^2 K^{*2} K^2 K^{*2}$

$= K^4 K^{*5}$ …………………… (β)

From (α) and (β) $K^*\in s(BQ)$.

- Theorem 5.Let $K\in L(H)$ be both self-adjoint, 2-self-adjoint, then $K\in s(BQ)$.

Proof

K being self-adjoint implies $K =K^*$ and 2-self-adjoint implies $K^2 =K^{*2}$.

Now;

$[K^{*2}K^2(K^*K)^2] K = K^{*2}K^2(K^*K)^2K$

$= K^{*2}K^2 K^{*2}K^2K$

$= K^2K^2 K^2K^2 K$

$= K^9$ …………………(ζ)

$K[(K^*K)^2K^{*2}K^2] = K(K^*K)^2K^{*2}K^2$

$= KK^{*2}K^2K^{*2}K^2$

$= K\,K^2K^2K^2K^2$

$= K^9$ …………………(ς)

From (ζ) and (ς) , $K \in s(BQ)$.

- Corollary 6. ([4] ,Corollary 2.6)Let $K \in L(H)$ be a self-adjoint operator, if β is real or pure imaginary number then βK is 2-self adjoint operator.
- Theorem 7.Let $K \in L(H)$ then the following holds.

  (i) $(K^* + K)^2 \in s(BQ)$.
  (ii) $(K^*K)^2 \in s(BQ)$.
  (iii) $K^{*2}K^2 \in s(BQ)$.
  (iv) $I + K^{*2}K^2 \in s(BQ)$ and $(I + K^*K)^2 \in s(BQ)$.

Proof

(i) Let $\gamma = (K^* + K)^2$

$\gamma = (K^* + K)^2 = (K^* + K)^{*2} = (K + K^*)^2 = \gamma$.

Thus γ is a 2-self-adjoint and combining Theorem 5 and Corollary 6 and letting β=1 , $\gamma = (K^* + K)^2 \in s(BQ)$.

(ii) $(K^*K)^2 = (K^*K)^{*2} = (KK^*)^2$

$(K^*K)^2$ is both self –adjoint and 2-self-adjoint hence $(K^*K)^2 \in s(BQ)$ by Theorem 5.

(iii) $K^{*2}K^2 = (K^{*2}K^2)^* = K^2K^{*2}$. K^2K^{*2} is a 2-self-adjoint hence $K^{*2}K^2 \in s(BQ)$.

(iv) $(I + K^{*2}K^2)^* = I + K^2K^{*2}$ and $(I + K^*K)^2 = (I + K^*K)^{*2} = (I + KK^*)^2$. Both $I + K^{*2}K^2$ and $(I + K^*K)^2$ are 2-self-adjoint and hence $I + K^{*2}K^2 \in s(BQ)$ and $(I + K^*K)^2 \in s(BQ)$.

- Theorem 8.Let K be be both self-adjoint and 2-self-adjoint. Then $M^*KM \in s(BQ)$.

Proof

$[(M^*KM)^{*2}(M^*KM)^2\{(M^*KM)^*M^*KM\}^2]M^*KM$

$= (M^*KM)^*(M^*KM)^*(M^*KM)(M^*KM)(M^*KM)^*(M^*KM)^*M^*KMM^*KMM^*KM$

$= (MKM^*)(MKM^*)(M^*KM)(M^*KM)(MKM^*)(MKM^*)M^*KMM^*KMM^*KM$

$= (M^*KM)^9$…………………………… (ħ)

$(M^*KM)[\{(M^*KM)^*(M^*KM)\}^2(M^*KM)^{*2}(M^*KM)^2]$

$= (M^*KM)(M^*KM)(M^*KM)(M^*KM)(M^*KM)(MKM^*)(MKM^*)(M^*KM)(M^*KM)$

$= (M^*KM)^9$…………………………… (ħ)

From (ħ) and (λ) , $M^*KM \in s(BQ)$.

Theorem 9.Let K be be both self-adjoint and 2-self-adjoint. Then $K^{-1} \in s(BQ)$.

Proof

We first verify that K^{-1} is a self-adjoint and 2-self-adjoint.

$(K^{-1})^* = (K^*)^{-1} = K^{-1}$ hence self-adjoint.

$(K^{-1})^{*2} = (K^{*2})^{-1} = (K^{-1})^2$ hence 2-self-adjoint. We now proceed to prove $K^{-1} \in s(BQ)$.

$[K^{*2}K^2(K^*K)^2]K = (K^{-1})^{*2}(K^{-1})^2((K^{-1})^*K^{-1})^2K^{-1}$

$= (K^{-1})^*(K^{-1})^*K^{-1}K^{-1}(K^{-1})^*(K^{-1})^*K^{-1}K^{-1}K^{-1}$

$= K^{-1}K^{-1}K^{-1}K^{-1}K^{-1}K^{-1}K^{-1}K^{-1}K^{-1}$

$= K^{-9}$………………………….(n)

$K[(K^*K)^2K^{*2}K^2] = K^{-1}((K^{-1})^*K^{-1})^2(K^{-1})^{*2}(K^{-1})^2$

$= K^{-1}(K^{-1})^{*2}(K^{-1})^2(K^{-1})^{*2}(K^{-1})^2$

$= K^{-1}(K^{-1})^*(K^{-1})^*K^{-1}K^{-1}(K^{-1})^*(K^{-1})^*K^{-1}K^{-1}$

$= K^{-1}K^{-1}K^{-1}K^{-1}K^{-1}K^{-1}K^{-1}K^{-1}K^{-1}$

$= K^{-9}$………………………….(p)

From (n) and (p) $K^{-1} \in s(BQ)$.

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